PAPER-DCQ1

Discrete–Continuous–Quantum Correspondence: A Phase–Encoded Six-Bit Embedding into $\mathrm{Gr}(3,6)$

Metric Compatibility, Finite Phase Sectors, and Berry–Chern Geometry

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Abstract

We construct a kinematic correspondence between a finite binary configuration space, a continuous Grassmannian geometry, and associated quantum-geometric carrier spaces. The core construction starts from the six-bit space

$$\mathcal{H}_6 = \{\pm 1\}^6,$$

with 64 configurations, and embeds it into the complex Grassmannian $\mathrm{Gr}(3,6)$ by a phase-encoding map.

A key point is that the phase assignment is chosen by a Gray-code rule. Thus changing one bit inside a bit-pair changes the corresponding phase by exactly $\pi/2$, while changing both bits changes it by π. This ensures metric compatibility: a pairwise Hamming-style metric on $\mathcal{H}_6$ is reproduced exactly by the Grassmannian principal-angle distance between the embedded 3-planes.

The Plücker/Fock carrier naturally associated with the Grassmannian embedding is

$$\Lambda^3(\mathbb{C}^6),$$

a 20-dimensional vector space. Separately, the three-pair structure of the six-bit space admits a threefold $\mathbb{C}^4$-tensor readout

$$(\mathbb{C}^4)^{\otimes 3},$$

whose fully symmetric and fully antisymmetric permutation sectors form the pure Bose–Fermi readout carrier

$$\mathcal{R}_{\mathrm{BF}} := \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4), \qquad \dim \mathcal{R}_{\mathrm{BF}} = 20 + 4 = 24.$$

This 24-dimensional readout space is not identified with $\Lambda^3(\mathbb{C}^6)$; it is a separate statistical readout layer.

The determinant line bundle on $\mathrm{Gr}(3,6)$ equips the construction with a Berry connection whose curvature represents an integral first Chern class. Moreover, the finite binary phase data embed into the fourth-root phase subgroup

$$\mu_4^3 \subset U(1)^3,$$

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so that continuous phase data restrict consistently to a finite discrete phase sector. The resulting framework is a kinematic foundation for a discrete–continuous–quantum correspondence. Dynamical, gravitational, and phenomenological interpretations are left to later work.

Paper-specific keywords: phase-encoded embedding, Gray-code phase encoding, metric compatibility, binary configuration space, Grassmannian geometry, Berry curvature, Chern class, finite phase sector, Bose–Fermi readout.

Geometric keywords: $\mathrm{Gr}(3,6)$, Plücker embedding, $\Lambda^3(\mathbb{C}^6)$, $(\mathbb{C}^4)^{\otimes 3}$, $\mu_4^3 \subset U(1)^3$, determinant line bundle.

Framework keywords: discrete–continuous–quantum correspondence, six-bit configuration space, finite phase encoding, pure Bose–Fermi readout.

Contents

1 Introduction

The purpose of this paper is to construct a precise kinematic bridge between three layers:

finite binary data, continuous Grassmannian geometry, quantum-geometric carrier spaces.

The finite layer is the six-bit configuration space

$$\mathcal{H}_6 = \{\pm 1\}^6.$$

The continuous layer is the Grassmannian

$$\mathrm{Gr}(3, 6),$$

whose points are complex 3-planes in $\mathbb{C}^6$. The quantum-geometric layer is represented first by the Plücker/Fock carrier

$$\Lambda^3(\mathbb{C}^6),$$

and separately by a pure Bose–Fermi readout carrier inside

$$(\mathbb{C}^4)^{\otimes 3}.$$

The key construction is a phase-encoded map

$$\iota : \mathcal{H}_6 \longrightarrow \mathrm{Gr}(3, 6).$$

The six bits are grouped into three pairs. Each pair determines one of four phase values:

$$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.$$

The phase assignment is chosen by a Gray-code convention: changing one bit in a pair changes the phase by $\pi/2$, while changing both bits changes it by π. Thus each pair supplies a finite fourth-root phase label, and the three pairs together determine an element of

$$\mu_4^3 \subset U(1)^3.$$

This finite phase sector is the discrete part of the construction.

The continuous part is obtained by assigning to each phase triple a 3-plane in $\mathbb{C}^6$, spanned by three orthonormal phase-block vectors. A central result of the paper is that the resulting embedding is metrically compatible: the discrete pairwise distance on $\mathcal{H}_6$ agrees exactly with the Grassmannian distance computed from principal angles.

The quantum-geometric interpretation has two distinct components. First, the Grassmannian embedding has the standard Plücker/Fock carrier

$$\Lambda^3(\mathbb{C}^6), \qquad \dim \Lambda^3(\mathbb{C}^6) = 20.$$

Second, the three-pair structure of the six-bit space naturally suggests a threefold four-state tensor carrier

$$(\mathbb{C}^4)^{\otimes 3}, \qquad \dim(\mathbb{C}^4)^{\otimes 3} = 64.$$

Inside this tensor carrier, the fully symmetric and fully antisymmetric subspaces define a pure statistical readout:

$$\mathcal{R}_{\mathrm{BF}} = \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4), \qquad \dim \mathcal{R}_{\mathrm{BF}} = 24.$$

This 24-dimensional readout space is not a decomposition of $\Lambda^3(\mathbb{C}^6)$. It is a separate readout layer.

Remark DCQ1-1.1 (Two distinct carrier spaces)**.** The distinction

$$\Lambda^3(\mathbb{C}^6) \quad \text{versus} \quad \mathcal{R}_{\mathrm{BF}}$$

is essential. The former is the 20-dimensional Plücker/Fock carrier of the Grassmannian embedding. The latter is the 24-dimensional pure Bose–Fermi readout carrier selected from $(\mathbb{C}^4)^{\otimes 3}$. The paper never identifies these two spaces.

The final geometric ingredient is Berry–Chern quantization. The Grassmannian carries a tautological subbundle

$$\mathcal{S} \longrightarrow \mathrm{Gr}(3,6),$$

and its determinant line bundle

$$\mathcal{L} = \det(\mathcal{S})$$

has a natural Chern connection. Its curvature represents the first Chern class of $\mathcal{L}$, yielding integral Berry fluxes over closed two-cycles. If one instead uses the Plücker hyperplane bundle $\det(\mathcal{S})^*$, the corresponding first Chern class changes sign; the integrality statement is unaffected.

1.1 Structure of the paper

Section 2 defines the phase-encoded embedding. Section 3 proves metric compatibility. Section 4 formulates finite phase-sector compatibility. Section 5 describes the Plücker/Fock carrier. Section 6 defines the separate pure Bose–Fermi readout carrier. Section 7 establishes Berry–Chern quantization. Section 8 clarifies the limited meaning of the entropy scales $\ln 64$ and $\ln 24$. The appendices provide representation-theoretic and finite holonomy details.

Remark DCQ1-1.2 (Scope)**.** This paper is kinematic. It does not derive dynamics, gravity, the Standard Model, black-hole entropy, or low-energy physical interactions. It constructs a finite-to-continuous geometric correspondence and records the carrier spaces and Berry–Chern structures naturally associated with that correspondence.

2 The Core Construction

Let

$$\mathcal{H}_6 = \{s = (s_1, \ldots, s_6) \mid s_i \in \{\pm 1\}\}$$

be the six-bit configuration space. We group the six bits into three ordered pairs:

$$(s_1, s_4), \qquad (s_2, s_5), \qquad (s_3, s_6).$$

For each bit-pair, define binary variables

$$b_i = \frac{1 - s_i}{2}, \qquad c_i = \frac{1 - s_{i+3}}{2}, \qquad i = 1, 2, 3.$$

Thus $b_i, c_i \in \{0, 1\}$.

Definition DCQ1-2.1 (Gray-coded phase labels)**.** For $s \in \mathcal{H}_6$, define

$$\theta_i(s) = \frac{\pi}{2}\big(b_i + 3c_i - 2b_i c_i\big), \qquad i = 1, 2, 3.$$

Equivalently, the phase assignment is given by the Gray-code table

(s_i, s_{i+3})	θ_i
$(+1, +1)$	0
$(-1, +1)$	$\frac{\pi}{2}$
$(-1, -1)$	π
$(+1, -1)$	$\frac{3\pi}{2}$

so that

$$\theta_i(s) \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}.$$

The four possible values of a pair (s_i, s_{i+3}) are therefore encoded by the four fourth-root phases

$$1,\ i,\ -1,\ -i.$$

The Gray-code convention ensures that changing exactly one bit in the pair changes the phase by $\pi/2$ modulo 2π, while changing both bits changes the phase by π.

Let $e_1, \dots, e_6$ be the standard basis of $\mathbb{C}^6$. For each $i = 1, 2, 3$, define

$$v_i(s) = e_i + e^{\mathbf{i}\theta_i(s)} e_{i+3}, \qquad \tilde{v}_i(s) = \frac{1}{\sqrt{2}} v_i(s).$$

Definition DCQ1-2.2 (Phase-encoded Grassmannian embedding)**.** The phase-encoded embedding is the map

$$\iota : \mathcal{H}_6 \longrightarrow \mathrm{Gr}(3, 6)$$

defined by

$$\iota(s) = \mathrm{span}_{\mathbb{C}}\{\tilde{v}_1(s), \tilde{v}_2(s), \tilde{v}_3(s)\}.$$

Because the vectors $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ have support on mutually disjoint coordinate pairs, they form an orthonormal frame of the corresponding 3-plane.

Proposition DCQ1-2.3 (Injectivity)**.** *The map $\iota : \mathcal{H}_6 \to \mathrm{Gr}(3, 6)$ is injective.*

Proof. Each bit-pair has four possible states, and the Gray-code definition of θ_i assigns these four states to the four distinct phases

$$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.$$

Thus the three bit-pairs determine a unique phase triple

$$(\theta_1, \theta_2, \theta_3).$$

This phase triple determines the three phase-block vectors

$$\tilde{v}_1, \tilde{v}_2, \tilde{v}_3.$$

Hence distinct bit configurations give distinct embedded 3-planes. □

3 Metric Compatibility

The embedding is designed so that discrete bit-pair changes match Grassmannian principal-angle distances.

Definition DCQ1-3.1 (Pairwise discrete distance)**.** For $s, s' \in \mathcal{H}_6$, define

$$h_i(s, s') = \frac{|s_i - s'_i| + |s_{i+3} - s'_{i+3}|}{2} \in \{0, 1, 2\}.$$

The pairwise discrete distance is

$$d_{\mathrm{pair}}(s, s') = \frac{\pi}{4}\sqrt{h_1(s, s')^2 + h_2(s, s')^2 + h_3(s, s')^2}.$$

Definition DCQ1-3.2 (Grassmannian principal-angle distance)**.** For $V, W \in \mathrm{Gr}(3,6)$, let

$$\alpha_1, \alpha_2, \alpha_3 \in [0, \pi/2]$$

be the principal angles between V and W. Define

$$\mathrm{dist}_{\mathrm{Gr}}(V, W) = \left(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\right)^{1/2}.$$

Theorem DCQ1-3.3 (Metric compatibility)**.** *For all $s, s' \in \mathcal{H}_6$,*

$$\mathrm{dist}_{\mathrm{Gr}}(\iota(s), \iota(s')) = d_{\mathrm{pair}}(s, s').$$

Proof. Let

$$V = \iota(s), \qquad W = \iota(s').$$

The orthonormal frames are

$$\tilde{v}_i = \frac{1}{\sqrt{2}}(e_i + e^{\mathbf{i}\theta_i} e_{i+3}), \qquad \tilde{v}_i' = \frac{1}{\sqrt{2}}(e_i + e^{\mathbf{i}\theta_i'} e_{i+3}).$$

The Gram matrix

$$G_{ij} = \langle \tilde{v}_i, \tilde{v}_j' \rangle$$

is diagonal:

$$G_{ij} = \delta_{ij} \frac{1 + e^{\mathbf{i}(\theta_i' - \theta_i)}}{2}.$$

Thus the singular values of G are

$$\left| \frac{1 + e^{\mathbf{i}\Delta_i}}{2} \right| = \left| \cos \frac{\Delta_i}{2} \right|, \qquad \Delta_i := \theta_i - \theta_i'.$$

Hence the i-th principal angle satisfies

$$\alpha_i = \frac{1}{2} |\Delta_i|_{\min},$$

where $|\Delta_i|_{\min}$ denotes the minimal representative modulo 2π in $[0, \pi]$.

By the Gray-code phase assignment, changing exactly one bit in the i-th pair changes θ_i by $\pi/2$ modulo 2π, while changing both bits changes θ_i by π. Therefore

$$|\Delta_i|_{\min} = \frac{\pi}{2} h_i(s, s').$$

Consequently,

$$\alpha_i = \frac{\pi}{4} h_i(s, s').$$

Taking the ℓ_2-norm gives

$$\mathrm{dist}_{\mathrm{Gr}}(\iota(s), \iota(s')) = \frac{\pi}{4} \sqrt{h_1(s, s')^2 + h_2(s, s')^2 + h_3(s, s')^2} = d_{\mathrm{pair}}(s, s').$$

□

Remark DCQ1-3.4 (Role of the Gray-code convention)**.** The metric theorem relies on the Gray-code phase assignment. A lexicographic assignment of the four pair states to the four phase values would not make every one-bit flip correspond to a $\pi/2$ phase increment. The Gray-code choice is therefore part of the metric structure of the embedding.

4 Finite Phase-Sector Compatibility

The phase-encoded construction provides a finite sampling of the continuous phase torus $U(1)^3$.
Let

$$\mu_4 := \{1, i, -1, -i\}$$

be the group of fourth roots of unity. Since each phase θ_i is one of

$$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$$

we have

$$e^{i\theta_i} \in \mu_4.$$

Definition DCQ1-4.1 (Finite phase-sector embedding)**.** Define

$$\Theta : \mathcal{H}_6 \longrightarrow \mu_4^3 \subset U(1)^3$$

by

$$\Theta(s) = \big(e^{i\theta_1(s)}, e^{i\theta_2(s)}, e^{i\theta_3(s)}\big).$$

Proposition DCQ1-4.2 (Finite phase-sector compatibility)**.** *The binary phase data factor through the finite subgroup*

$$\mu_4^3 \subset U(1)^3.$$

Thus continuous $U(1)^3$-phase data restricts on the embedded binary sector to finite μ_4^3-valued phase data.

Proof. For each i, the phase $\theta_i(s)$ belongs to

$$\left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}.$$

Therefore

$$e^{i\theta_i(s)} \in \mu_4.$$

For three independent pairs,

$$\Theta(s) \in \mu_4^3.$$

Since μ_4^3 is a finite subgroup of $U(1)^3$, the assertion follows. □

Proposition DCQ1-4.3 (Pair transitions as finite phase increments)**.** *For each bit-pair (s_i, s_{i+3}), a one-bit change corresponds to a nearest-neighbor transition among the four fourth-root phase labels. A two-bit change corresponds to the opposite phase label. Hence pairwise transition data are represented by finite phase increments in μ_4.*

Proof. By the Gray-code table, the four pair states form a cycle

$$(+,+) \longrightarrow (-,+) \longrightarrow (-,-) \longrightarrow (+,-) \longrightarrow (+,+),$$

corresponding to phase increments by $\pi/2$. Thus one-bit changes move to adjacent fourth-root phases, while changing both bits gives a phase difference of π. Applying this to all three pairs gives finite transition data in μ_4^3. □

Remark DCQ1-4.4 (No categorical equivalence claim)**.** This paper does not claim an equivalence

$$\mathbf{Loc}_{\mathbb{Z}_2}(\mathcal{M}_{\mathrm{disc}}) \simeq \mathbf{Loc}_{U(1)}(\mathcal{M}_{\mathrm{cont}}).$$

Such a statement would require much stronger hypotheses and is generally not justified by the present construction. The result here is the finite phase-sector embedding

$$\mathcal{H}_6 \longrightarrow \mu_4^3 \subset U(1)^3.$$

5 The Plücker/Fock Carrier

The Grassmannian embedding has a natural projective quantum carrier given by the Plücker construction.

Let

$$V = \iota(s) \in \mathrm{Gr}(3,6)$$

be represented by the orthonormal frame

$$(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3).$$

Define

$$|\Psi(V)\rangle := \tilde{v}_1 \wedge \tilde{v}_2 \wedge \tilde{v}_3 \in \Lambda^3(\mathbb{C}^6).$$

Definition DCQ1-5.1 (Plücker/Fock map)**.** The Plücker/Fock map is

$$\mathcal{Q}_{\mathrm{Pl}} : \mathrm{Gr}(3,6) \longrightarrow \mathbb{P}(\Lambda^3\mathbb{C}^6), \qquad V \longmapsto [|\Psi(V)\rangle].$$

Changing the orthonormal frame of V by an element of $U(3)$ changes the wedge by a determinant phase. Hence the projective ray is well-defined.

Proposition DCQ1-5.2 (Dimension of the Plücker carrier)**.** *The Plücker/Fock carrier has dimension*

$$\dim \Lambda^3(\mathbb{C}^6) = \binom{6}{3} = 20.$$

Proof. This is the standard dimension formula for exterior powers:

$$\dim \Lambda^k(\mathbb{C}^n) = \binom{n}{k}.$$

Taking $n = 6$ and $k = 3$ gives

$$\dim \Lambda^3(\mathbb{C}^6) = \binom{6}{3} = 20.$$

□

Remark DCQ1-5.3 (Relation to $\mathrm{Spin}(6) \simeq SU(4)$)**.** There is a standard isomorphism

$$\mathrm{Spin}(6) \simeq SU(4).$$

Under this isomorphism, the complexified vector representation of Spin(6) may be identified with the second exterior power of the fundamental representation of $SU(4)$:

$$\mathbb{C}^6 \simeq \Lambda^2(\mathbb{C}^4).$$

Consequently, the Plücker/Fock carrier admits an $SU(4)$-equivariant description

$$\Lambda^3(\mathbb{C}^6) \simeq \Lambda^3(\Lambda^2\mathbb{C}^4).$$

This remains a 20-dimensional carrier. In the $SU(4)$ language, the middle exterior power of the six-dimensional vector representation is more naturally related to the two 10-dimensional chiral components, schematically

$$\Lambda^3(\mathbb{C}^6) \simeq \mathrm{Sym}^2(\mathbb{C}^4) \oplus \mathrm{Sym}^2((\mathbb{C}^4)^*),$$

up to the usual convention for the two chiral spinor representations.

In particular,

$$\Lambda^3(\mathbb{C}^6) \not\simeq \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4).$$

The latter is the separate 24-dimensional pure Bose–Fermi readout carrier inside $(\mathbb{C}^4)^{\otimes 3}$, not the Plücker carrier.

6 The Pure Bose–Fermi Readout Carrier

The six-bit space has three bit-pairs, each with four possible phase states. This motivates a separate threefold tensor readout.

Let

$$W := \mathbb{C}^4.$$

The three-pair readout carrier is

$$W^{\otimes 3} = (\mathbb{C}^4)^{\otimes 3}.$$

Its dimension is

$$\dim W^{\otimes 3} = 4^3 = 64,$$

matching

$$|\mathcal{H}_6| = 64.$$

Definition DCQ1-6.1 (Pair-state readout basis)**.** Choose a basis

$$f_0, f_1, f_2, f_3$$

of $W = \mathbb{C}^4$, labelled by the four phase values

$$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.$$

Each bit-pair is assigned to one of these four basis vectors. The full six-bit configuration is assigned to a basis tensor in

$$W^{\otimes 3}.$$

Thus there is a natural finite readout map

$$\mathcal{H}_6 \longrightarrow (\mathbb{C}^4)^{\otimes 3}.$$

The symmetric group S_3 acts on $W^{\otimes 3}$ by permuting the three tensor factors. The two extreme permutation-symmetry sectors are

$$\mathrm{Sym}^3(W), \qquad \Lambda^3(W).$$

Definition DCQ1-6.2 (Pure Bose–Fermi readout carrier)**.** The pure Bose–Fermi readout carrier is

$$\mathcal{R}_{\mathrm{BF}} := \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4).$$

Proposition DCQ1-6.3 (Dimension of the pure Bose–Fermi carrier)**.** *The readout carrier $\mathcal{R}_{\mathrm{BF}}$ has dimension*

$$\dim \mathcal{R}_{\mathrm{BF}} = 20 + 4 = 24.$$

Proof. For $W = \mathbb{C}^4$,

$$\dim \mathrm{Sym}^3(W) = \binom{4+3-1}{3} = \binom{6}{3} = 20.$$

Also,

$$\dim \Lambda^3(W) = \binom{4}{3} = 4.$$

Therefore

$$\dim \mathcal{R}_{\mathrm{BF}} = 20 + 4 = 24.$$

□

Remark DCQ1-6.4 (No identification with the Plücker carrier)**.** One must not write

$$\Lambda^3(\mathbb{C}^6) \simeq \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4).$$

The left-hand side has dimension 20, whereas the right-hand side has dimension 24. Therefore $\mathcal{R}_{\mathrm{BF}}$ is a separate readout carrier inside $(\mathbb{C}^4)^{\otimes 3}$, not a decomposition of $\Lambda^3(\mathbb{C}^6)$.

6.1 Schur–Weyl perspective

The full tensor carrier contains more than the pure Bose–Fermi readout. Schur–Weyl theory gives a decomposition of the schematic form

$$W^{\otimes 3} \simeq \mathrm{Sym}^3(W) \oplus \left(S_{(2,1)}W\right)^{\oplus 2} \oplus \Lambda^3(W),$$

where $S_{(2,1)}W$ denotes the mixed-symmetry Schur functor.

For $W = \mathbb{C}^4$, the dimensions are

$$64 = 20 + 40 + 4.$$

Thus $\mathcal{R}_{\mathrm{BF}}$ selects the two pure permutation statistics sectors:

$$\mathcal{R}_{\mathrm{BF}} = \mathrm{Sym}^3(W) \oplus \Lambda^3(W),$$

while the mixed-symmetry sector is not included in the pure readout.

Principle DCQ1-6.5 (Pure-statistics readout layer)**.** The 24-dimensional space $\mathcal{R}_{\mathrm{BF}}$ is best understood as a pure-statistics readout layer associated with the three-pair tensor carrier $W^{\otimes 3}$. It is not the primary Grassmannian quantum carrier.

Remark DCQ1-6.6 (Why the $20 : 4$ asymmetry is not paradoxical)**.** The dimensions of the fully symmetric and fully antisymmetric sectors of $W^{\otimes 3}$ need not be equal. They are governed by different combinatorial rules:

$$\dim \mathrm{Sym}^3(W) = \binom{\dim W + 3 - 1}{3}, \qquad \dim \Lambda^3(W) = \binom{\dim W}{3}.$$

For $W = \mathbb{C}^4$, these give 20 and 4, respectively. The asymmetry reflects the fact that symmetric tensors allow repeated labels, whereas exterior tensors do not.

7 Berry–Chern Geometry

The Grassmannian carries natural vector bundles and characteristic classes.

Let

$$\mathcal{S} \longrightarrow \mathrm{Gr}(3,6)$$

be the tautological rank-3 subbundle, and define its determinant line bundle

$$\mathcal{L} := \det(\mathcal{S}).$$

Definition DCQ1-7.1 (Berry connection)**.** The Chern connection on the Hermitian line bundle $\mathcal{L}$ is called the Berry connection in this setting. Its curvature is denoted by

$$\Omega.$$

Theorem DCQ1-7.2 (Berry–Chern quantization)**.** *The Berry curvature represents the first Chern class:*

$$\left[\frac{\Omega}{2\pi}\right] = c_1(\mathcal{L}) \in H^2(\mathrm{Gr}(3,6);\mathbb{Z}).$$

Consequently, for every closed oriented 2-cycle

$$\Sigma \subset \mathrm{Gr}(3,6),$$

one has

$$\frac{1}{2\pi}\int_\Sigma \Omega \in \mathbb{Z}.$$

Proof. By Chern–Weil theory, the curvature of a unitary connection on a Hermitian line bundle represents its first Chern class. With the chosen normalization,

$$\left[\frac{\Omega}{2\pi}\right] = c_1(\mathcal{L}) \in H^2(\mathrm{Gr}(3,6);\mathbb{Z}).$$

Evaluating this integral cohomology class on the homology class of a closed oriented 2-cycle Σ gives

$$\langle c_1(\mathcal{L}), [\Sigma]\rangle = \frac{1}{2\pi}\int_\Sigma \Omega \in \mathbb{Z}.$$

□

Remark DCQ1-7.3 (Sign convention)**.** The sign convention for

$$\mathcal{L} = \det(\mathcal{S})$$

fixes the sign of Ω. If one instead uses the Plücker hyperplane bundle

$$\det(\mathcal{S})^*,$$

the corresponding Chern class changes sign. The integrality statement itself is unaffected.

Remark DCQ1-7.4 (Geometric meaning)**.** The quantization statement is a standard Chern-class integrality statement. It does not require a separate operator quantization postulate. It follows from the existence of the determinant line bundle and its unitary connection.

8 Entropy Scales and Their Limited Interpretation

The construction contains two different finite counts.

First, the discrete configuration space has

$$|\mathcal{H}_6| = 2^6 = 64$$

states. The corresponding maximal discrete entropy scale is

$$S_{\mathrm{disc}} = \ln 64 = 6\ln 2.$$

Second, the pure Bose–Fermi readout carrier has dimension

$$\dim \mathcal{R}_{\mathrm{BF}} = 24,$$

giving a readout entropy scale

$$S_{\mathrm{BF}} = \ln 24.$$

Remark DCQ1-8.1 (No black-hole entropy derivation)**.** The appearance of $\ln 24$ in this paper is a representation-theoretic readout count. It is not a derivation of black-hole entropy. Any relation to black-hole microstate counting or logarithmic entropy corrections requires additional physical dynamics and is not established in DCQ1.

Remark DCQ1-8.2 (Two-layer interpretation)**.** The scale $\ln 64$ belongs to the full discrete six-bit layer, whereas $\ln 24$ belongs to the pure Bose–Fermi readout layer. The relation between these two scales may be interpreted as a structural compression from a finite phase-labelled configuration space to a pure-statistics readout layer, but no thermodynamic law is derived here.

Remark DCQ1-8.3 (Optional contextual use)**.** References to black-hole entropy, if included, should be read only as contextual motivation for why logarithms of finite state counts are interesting in quantum gravity. No black-hole entropy formula is used or derived in this paper.

9 Conclusion and Outlook

This paper constructed a phase-encoded embedding

$$\iota : \mathcal{H}_6 \longrightarrow \mathrm{Gr}(3,6)$$

from the 64-element six-bit configuration space into the Grassmannian of 3-planes in $\mathbb{C}^6$. The phase assignment is Gray-coded, so that one-bit changes inside a pair correspond to $\pi/2$ phase increments. This choice makes the embedding metrically compatible: the pairwise Hamming-style distance on $\mathcal{H}_6$ is exactly reproduced by the Grassmannian principal-angle distance.

The primary Grassmannian quantum carrier is the Plücker/Fock space

$$\Lambda^3(\mathbb{C}^6), \qquad \dim \Lambda^3(\mathbb{C}^6) = 20.$$

Separately, the three-pair structure of the binary configuration space admits a threefold four-state tensor readout

$$(\mathbb{C}^4)^{\otimes 3}.$$

Its fully symmetric and fully antisymmetric permutation sectors define the pure Bose–Fermi readout carrier

$$\mathcal{R}_{\mathrm{BF}} = \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4), \qquad \dim \mathcal{R}_{\mathrm{BF}} = 24.$$

This readout carrier is not a decomposition of $\Lambda^3(\mathbb{C}^6)$.

The paper also established finite phase-sector compatibility:

$$\mathcal{H}_6 \longrightarrow \mu_4^3 \subset U(1)^3.$$

This replaces any stronger claim of a full categorical equivalence between discrete and continuous local systems. The result is a finite phase embedding, not a complete holographic equivalence.

Finally, the determinant line bundle on $\mathrm{Gr}(3,6)$ supplies a Berry connection whose curvature represents an integral first Chern class. This gives a standard Berry–Chern quantization mechanism within the Grassmannian geometry.

Future work may add dynamics, path integrals, effective readout maps, or physical applications. Such later developments should be built on top of the kinematic structures established here rather than assumed as part of DCQ1 itself.

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A Representation-Theoretic Details of the Pure Bose–Fermi Readout

Let
$$W = \mathbb{C}^4.$$
The full threefold tensor carrier is
$$W^{\otimes 3}.$$
It has dimension
$$\dim W^{\otimes 3} = 4^3 = 64.$$

The symmetric group S_3 acts on $W^{\otimes 3}$ by permuting tensor factors. The fully symmetric subspace is
$$\mathrm{Sym}^3(W),$$
and the fully antisymmetric subspace is
$$\Lambda^3(W).$$

Their dimensions are
$$\dim \mathrm{Sym}^3(\mathbb{C}^4) = \binom{4+3-1}{3} = 20,$$
and
$$\dim \Lambda^3(\mathbb{C}^4) = \binom{4}{3} = 4.$$
Thus
$$\dim \left(\mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4)\right) = 24.$$

The mixed-symmetry sector accounts for the remaining dimension:
$$64 - 24 = 40.$$
Schematically,
$$W^{\otimes 3} \simeq \mathrm{Sym}^3(W) \oplus \left(S_{(2,1)}W\right)^{\oplus 2} \oplus \Lambda^3(W).$$

Remark DCQ1-A.1. This appendix justifies the $20 + 4 = 24$ count of the pure Bose–Fermi readout carrier. It does not claim that this space is $\Lambda^3(\mathbb{C}^6)$.

B Finite Phase-Sector Restriction

For each bit-pair (s_i, s_{i+3}), define

$$b_i = \frac{1 - s_i}{2}, \qquad c_i = \frac{1 - s_{i+3}}{2}, \qquad i = 1, 2, 3.$$

Thus $b_i, c_i \in \{0, 1\}$. The Gray-coded phase assignment is

$$\theta_i = \frac{\pi}{2}\big(b_i + 3c_i - 2b_ic_i\big).$$

Equivalently,

(s_i, s_{i+3})	(b_i, c_i)	θ_i	$e^{\mathbf{i}\theta_i}$
$(+1, +1)$	$(0, 0)$	0	1
$(-1, +1)$	$(1, 0)$	$\frac{\pi}{2}$	$\mathbf{i}$
$(-1, -1)$	$(1, 1)$	π	-1
$(+1, -1)$	$(0, 1)$	$\frac{3\pi}{2}$	$-\mathbf{i}$

Therefore each bit-pair determines a fourth-root phase:

$$e^{\mathbf{i}\theta_i} \in \mu_4 = \{1, \mathbf{i}, -1, -\mathbf{i}\}.$$

For three independent bit-pairs, the full phase label lies in

$$\mu_4^3 \subset U(1)^3.$$

Proposition DCQ1-B.1 (Finite phase sampling)**.** *The Gray-coded phase assignment defines a finite phase-sector map*

$$\Theta : \mathcal{H}_6 \longrightarrow \mu_4^3 \subset U(1)^3$$

by

$$\Theta(s) = \big(e^{\mathbf{i}\theta_1(s)}, e^{\mathbf{i}\theta_2(s)}, e^{\mathbf{i}\theta_3(s)}\big).$$

Thus the binary configuration space is realised as a finite fourth-root phase sampling of the continuous phase torus $U(1)^3$.

Proof. For each pair (s_i, s_{i+3}), the Gray-coded formula gives one of the four values

$$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.$$

Hence

$$e^{\mathbf{i}\theta_i(s)} \in \mu_4.$$

Since the three bit-pairs are independent, the full phase triple belongs to

$$\mu_4^3.$$

Therefore

$$\Theta(\mathcal{H}_6) \subset \mu_4^3 \subset U(1)^3.$$

□

Proposition DCQ1-B.2 (Gray-code transition compatibility)**.** *For each bit-pair (s_i, s_{i+3}), changing exactly one bit changes the phase label by $\pi/2$ modulo 2π. Changing both bits changes the phase label by π modulo 2π. Hence the finite phase-sector map is compatible with the pairwise transition structure used in the metric compatibility theorem.*

Proof. From the Gray-code table, the four pair states form the cycle

$$(+1,+1) \longrightarrow (-1,+1) \longrightarrow (-1,-1) \longrightarrow (+1,-1) \longrightarrow (+1,+1),$$

with corresponding phase labels

$$0 \longrightarrow \frac{\pi}{2} \longrightarrow \pi \longrightarrow \frac{3\pi}{2} \longrightarrow 0.$$

Adjacent states in this cycle differ by exactly one bit and by a phase increment of $\pi/2$ modulo 2π. Opposite states in the cycle differ by two bits and by a phase increment of π modulo 2π. Thus the finite phase-sector map preserves the intended pairwise transition structure. $\square$

Remark DCQ1-B.3 (Finite restriction, not categorical equivalence)**.** This appendix establishes only the finite phase-sector restriction

$$\mathcal{H}_6 \longrightarrow \mu_4^3 \subset U(1)^3.$$

It does not assert an equivalence between categories of discrete and continuous local systems. The result is a finite, Gray-code-compatible sampling of the continuous phase torus, not a full discrete–continuous categorical equivalence.

Remark DCQ1-B.4 (Relation to metric compatibility)**.** The same Gray-code convention is responsible for the metric compatibility proved in the main text. Because one-bit changes correspond to $\pi/2$ phase increments and two-bit changes correspond to π phase increments, the principal-angle distance between the embedded Grassmannian 3-planes exactly reproduces the pairwise discrete distance on $\mathcal{H}_6$.

C Minimal Binary Realisation and the Entropy Unit $\ln 2$

A finite distinguishable alphabet is a finite set

$$\mathcal{A} = \{a_1, \dots, a_N\}$$

of mutually distinguishable labels. It is nontrivial if $N \geq 2$.

Proposition DCQ1-C.1 (Minimal nontrivial finite distinguishability is binary)**.** *Any minimal nontrivial finite distinguishable alphabet has cardinality*

$$|\mathcal{A}| = 2.$$

Proof. A nontrivial alphabet must contain at least two elements. The set $\{0, 1\}$ realizes this lower bound. Hence the minimal nontrivial finite alphabet is binary. □

Lemma DCQ1-C.2 (Entropy bound for a binary variable)**.** *Let a binary variable have probabilities p and $1-p$. Its Shannon entropy*

$$S(p) = -p \ln p - (1-p) \ln(1-p)$$

satisfies

$$S(p) \leq \ln 2,$$

with equality if and only if $p = 1/2$.

Proof. The derivative is

$$S'(p) = \ln \frac{1-p}{p},$$

so the unique critical point is $p = 1/2$. The second derivative is negative:

$$S''(p) = -\frac{1}{p} - \frac{1}{1-p} < 0.$$

Thus the critical point is the unique maximum, and

$$S(1/2) = \ln 2.$$

□

For six binary degrees of freedom,

$$|\mathcal{H}_6| = 2^6 = 64,$$

and the corresponding maximal discrete entropy scale is

$$\ln 64 = 6 \ln 2.$$

The readout scale $\ln 24$ belongs to the separate pure Bose–Fermi readout carrier

$$\mathcal{R}_{\mathrm{BF}}, \qquad \dim \mathcal{R}_{\mathrm{BF}} = 24.$$

It is not derived from minimal binary distinguishability alone.

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End of Paper